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Submillimetre Solar Radiation at Sea Level

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Submillimetre solar radiation at sea level

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The detection of solar radiation in the 860 μm window at a sea-level site (London, England) is reported. The measurements give a value of $7\cdot2 \pm 2\cdot0$ dB/km of standard atmosphere for the minimum attenuation at the centre of the window.

The possibility of submillimetre astronomy from low altitude sites is discussed. We conclude that most of the world's observatories in temperate or cool climates could make worthwhile measurements in the 730 and 860 μm windows for a limited period each year. Measurements at 350 and 450 μm should be possible from a more limited number of sites in cold climates. Low temperature seems to be the most important criterion in choosing a site with low precipitable water-vapour content.

1. Introduction

This paper reports the detection at a sea-level site of appreciable amounts of submillimetre radiation, and also discusses the suitability of various sites for submillimetre astronomical observations.

There have already been several reports of the detection of submillimetre solar radiation, but these have all been from high altitude sites. In 1957 Gebbie used a two-beam Michelson interferometer sited at the Jungfraujoch Observatory (3450 m altitude). This work, carried out at relatively high resolution, showed the presence of large amounts of submillimetre radiation transmitted through the windows formed by the pure rotational absorption of the water vapour in the Earth's atmosphere. This work was confirmed at higher altitudes by Aberkov, Anikin, Ryadov & Furashov (1964) working in the Pamirs at 3860 m and by Farmer & Key (1965) working in the Andes at 5200 m. Both these groups used grating spectrometers. More recently Gaitskell & Gear (1966) have detected submillimetre radiation at a considerably lower altitude at the Pic-du-Midi Observatory (2880 m), using a Fabry–Pérot interferometer of the type described by Renk & Genzel (1962). Gaitskell & Gear also detected lunar radiation and made absolute (if somewhat imprecise) determinations of the moon's temperature at wavelengths of 740 and 900 μm.

We shall restrict our discussion to wavelengths above 300 μm. Just below this wavelength a number of water-vapour lines form an intense absorption region covering about 50 μm and this produces a convenient demarcation between the windows above 300 μm and the less transparent windows below 200 μm wavelength. We shall also consider the wide window which is centred above a 1 mm wavelength and is bounded by the transitions 4_0–5_{-4} at 926 μm and 2_2–3_{-2} at 1626 μm. We include this window in the survey for two experimental reasons. First, as with the submillimetre windows, the atmospheric absorption within this window is very important, and in any astronomical measurement careful methods must be devised to eliminate its effect; secondly, this window is similar

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to the submillimetre windows in that in both cases broad-band bolometric detecters are more convenient than narrow-band microwave detectors. The submillimetre windows, together with the window centred at about 1300 μm, therefore form a convenient group which will be considered together.

2. Experimental technique

The telescope used for the observations was similar in most respects to the telescope described by Gaitskell & Gear, but was mounted on the roof of the Physics Department at Queen Mary College. The elevation of this instrument was in fact about 50 m above sea level, but simple calculations show that the extra 50 m between the site and sea level would make little or no difference in the observed total attenuation, so that the title given to this paper is justifiable if not strictly accurate.

To reduce noise from sky and instrument radiation at short submillimetre wavelengths, 0·2 mm of soot-impregnated polyethylene and 1·5 mm of the alkali halide filters (described by Yamada, Mitsusishi & Yoshinaga 1962) were used as filters. Probably as a result of the insufficient structural support within the building, the pointing accuracy of the telescope was not so great as that used at the Pic-du-Midi Observatory. A random r.m.s. error of about ± 4 min arc was found for both hour-angle and declination scales. However, the alinement between the radio axis and the axes of the visual finder telescopes was considerably better than this, so that relatively good setting of the radio axis on the centre of the solar disk was achieved by making fine adjustments to the declination setting and to the rate of the hour-angle drive throughout any spectral scan.

Because of the fairly low spectral resolution (*ca.* 0·5 cm^{-1} at 800 μm), we decided to use the Fabry–Pérot instrument entirely as a spectrally scanning instrument rather than as a fixed etalon. In this way the magnitude of the effects due to the wings of the spectral response function are far more easy to investigate, although of course the method has a disadvantage in that it does not eliminate the effects of spectral variation in the intensity of the background atmospheric radiation.

Both Golay and cooled indium antinomide detectors were used in the measurements. The indium antinomide detector was kept at 4 °K at the end of a polished light pipe which was coaxial with the axis of the Cassegrain telescope. We hope to give a more detailed description of the use of this detector in a later publication dealing with lunar millimetre and submillimetre results. Unfortunately the cooled detector did not come into operation until May 1966 when the driest weather had past. Submillimetre radiation was however still then readily detectable.

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3. Results

By means of the method described above, Fabry–Pérot spectral trace recordings were made on a number of clear or nearly clear days in the spring of 1966. On most of these days the spectral trace showed a definite peak corresponding to a wavelength of about 860 μm. With sufficiently low water-vapour concentrations, such a peak would in fact be expected from radiation passing through the atmosphere in the window delineated by the absorptions 3_1–4_{-3} (795 μm) and 4_0–5_{-4} (926 μm). On several occasions the output trace showed a slight rise corresponding to a wavelength of about 730 μm which would be expected as a result of transmission through the window defined by the lines 3_3–4_{-1} (672 μm) and 3_1–4_{-3} (795 μm). However, detection of radiation through this window was not certain, and the observations did not therefore warrant any numerical analysis. Throughout the work there was no indication of atmospheric transmission at any shorter wavelengths.

Two typical traces of the detector output during spectral scans are shown in figures 1 and 2. Golay and cooled indium antinomide detectors were used respectively for these measurements. It can be clearly seen that, although the scan using the Golay cell has much larger noise level, both scans show the same general form. Before proceeding to a detailed analysis of the attenuation of the 860 μm window, we note a number of more minor interesting features which are common to both traces and to most of the other recordings we have made.

(1) There is a noticeable dip in the recorded spectrum at about 350 μm, and we attribute this to changes with wavelength of the intensity of atmospheric radiation. Radiosonde measurements from nearby stations on both the days represented in figures 1 and 2 show a temperature decreasing with height for the first few thousand metres of the atmosphere. Provided the optical depth of the atmosphere τ_λ at these wavelengths is considerably greater than unity (say $\tau_\lambda > 5$), then we should expect no solar radiation to be recorded. The radiation from the atmosphere will however correspond to a temperature at some weighted mean depth d_λ which approximates to the mean height from which the photons of the wavelength λ reach the detector. In wavelength regions (such as 360 μm) where the absorption is low compared with nearby wavelengths (300 and 400 μm), d_λ will be relatively large (*ca.* 500 m). The corresponding intensities will be relatively low because of the negative temperature gradient of the atmosphere, and this is in fact what is observed. Similar effects have been reported by Williams & Chang (1966) who observed from Kitt Peak National Observatory (2064 m) using an interferometric technique.

(2) The trace height shows a slight peak at about 500 μm. In terms of the above discussion, this is again to be expected because of the very intense 1_{-1}–1_1 transition centred close to this wavelength (538 μm). This transition in fact dominates the submillimetre water-vapour spectrum. Both this and the effect at 360 μm could of course be given a quantitative theoretical treatment, rather similar to that used in the discussions of stellar atmospheres in which local thermodynamic equilibrium is assumed.

(3) The trace rises to a maximum intensity at a wavelength of about 1100 μm and then falls slowly at longer wavelengths as would be expected from the competition between the Rayleigh–Jeans law of the source and the even more rapid decrease with wavelength of the atmospheric attenuation.

(4) The traces all show a minimum intensity at about 1626 μm. This is presumably due to the 2_2–3_{-2} water-vapour absorption line centred at this wavelength. It should be realized, however, that the instrument contains no device for suppression of second-order spectra, so it is likely that the rise of the trace at about 1750 μm is partly due to the beginning of the second-order peak of radiation of wavelength centred at 860 μm.

We now proceed to a more detailed analysis of the radiation transmitted by the window centred at 860 μm. There seems little doubt that the observed peak on the detector trace

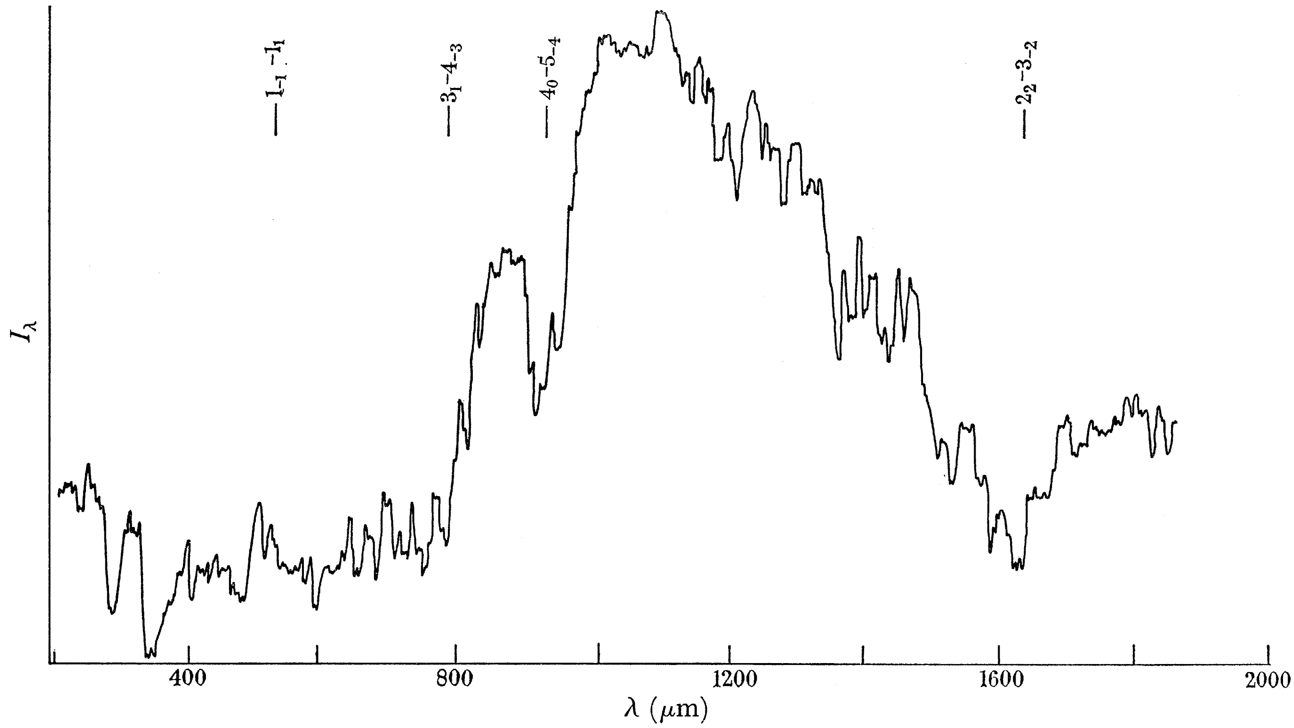

FIGURE 1. Fabry–Pérot interferometer spectral scan of solar radiation recorded 10.00 G.M.T. 25 March 1966, by means of a Golay cell. There were no clouds covering the Sun throughout the scan. The vertical scale has been expanded so that unit flux incident on the telescope is represented by the same vertical interval as in figure 2. Otherwise the figure is a direct copy of the detector output.

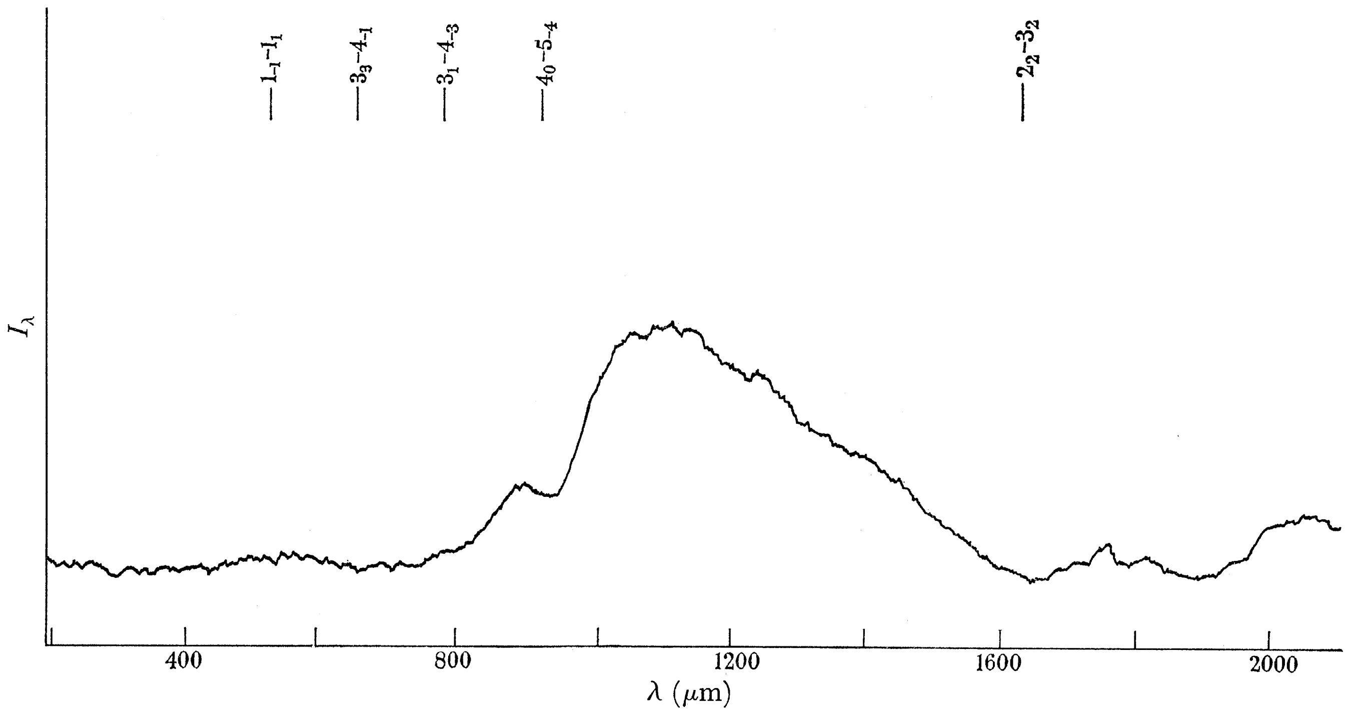

FIGURE 2. Fabry–Pérot interferometer spectral scan of solar radiation recorded 11.40 G.M.T. 10 May 1966, by means of the cooled InSb detector. Light clouds covered the Sun for several short intervals during the portion of the scan at wavelengths above 1200 μm. The figure is a direct copy of the output trace of the detector.

is due to radiation within the window between the lines at 795 and 927 μm. The wavelength for the centre of the peak measured by the scale of the interferometer corresponds well with the centre of the window. Also measurements made on humid days show the peak to be completely absent.

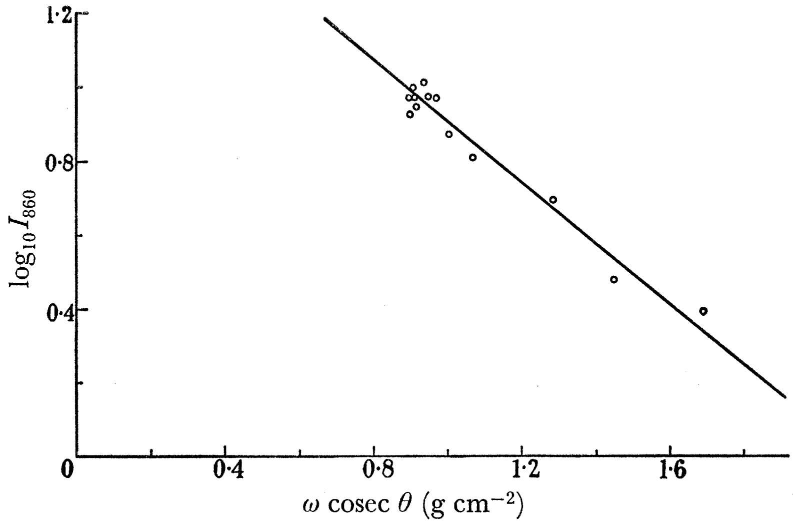

FIGURE 3. Intensity measurements at 860 μm made throughout 25 March 1966, as a function of water-vapour path-length. Each point represents the intensity measurement of a separate spectral scan.

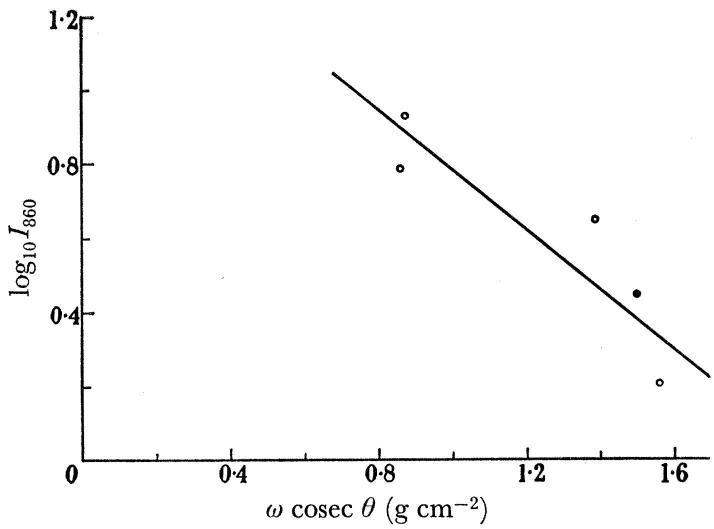

FIGURE 4. Intensity measurements at 860 μm made on various days in March and May 1966. The closed circle represents a measurement made with the InSb detector. Open circles all represent measurements made with the Golay detector.

The results of measurements of the radiation intensity within this window are shown in figures 3 and 4. In each case the peak height of the recording was taken as a measure of intensity, while radiosonde data from the nearby meteorological station at Crawley were used to determine ω_0, the precipitable water in the atmosphere.

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Figure 3 shows the results of thirteen spectral scans made on 25 March 1966 recorded over a 6 h period centred at about noon. That day was practically clear; there was relatively low precipitable water (5·6 mm), and the recorded intensity varied fairly smoothly throughout the period of observation. The variation of the cosecant of the solar elevation ($\operatorname{cosec}\theta$) throughout the measurements allows the atmospheric attenuation at the centre of the window to be determined. This measurement is of considerable importance to ground-based submillimetre astronomy, and there are so far few accurate measures of the attenuation. The problem of determining this attenuation from the measurements described above has therefore been considered in some detail. For monochromatic radiation of wavelength λ, the specific intensity at ground level I_λ is related to the intensity $I_{0\lambda}$ outside the atmosphere by the relation

$$I_\lambda = I_{0\lambda} \exp\left\{-\operatorname{cosec}\theta \int_0^\infty k_\lambda \rho \, \mathrm{d}h\right\}, \tag{1}$$

where k_λ is the mass absorption coefficient and ρ the density of water vapour at height h. If we assume for the moment that k_λ is independent of temperature and pressure, then equation (1) can be rewritten in terms of the precipitable water vapour ω:

$$I_\lambda = I_{0\lambda} \exp\{-\omega k_\lambda \operatorname{cosec}\theta\}. \tag{2}$$

The measurements made on 26 March 1966 showed the same asymmetry about noon as that reported previously by Bastin, Gear, Jones, Smith & Wright (1964), so that we have assumed ω varies linearly with the time t measured from solar transit:

$$\omega = \omega_0(1+\beta t). \tag{3}$$

Equations (2) and (3) give

$$\log_{10} I_\lambda = \log_{10} I_{0\lambda} - \omega_0(1+\beta t)\, k_\lambda \operatorname{cosec}\theta \log_{10} \mathrm{e}. \tag{4}$$

A value of $\beta = +0·06$ g cm^{-2} h^{-1} was found by inspection, and the values of $\omega_0 \operatorname{cosec}\theta$ were multiplied by $(1+\beta t)$ to give the water-vapour path-length, the quantity which is represented as the abscissa of figures 3 and 4.

For convenience atmospheric attenuations are usually stated in decibels (dB) per kilometre of a standard atmosphere. Since a kilometre of standard atmosphere contains a precipitable water-vapour path-length of 0·75 g cm^{-2}, the attenuation at 860 μm is readily found from figure 2. From this figure we deduce a value of $6·4 \pm 0·4$ dB/km for an atmosphere containing 7·5 g m^{-3}. The relative closeness of the points to a straight line shows that the assumption that the atmospheric water-vapour content increases linearly with time is a good first approximation. The cluster of points at the top right-hand end of the graph represents measurements made between 10.00 and 11.30 G.M.T. when the opposing effects of increasing water vapour and decreasing zenith angle virtually compensated for each other so as to give an approximately constant water-vapour path-length. The scatter of the points indicates the random errors introduced by noise in the measuring systems and short-term fluctuations in the water-vapour path-length, and it is on these random errors that the quoted error of $\pm 0·4$ dB/km is based. This error does not, however, take into account the rather more serious systematic error which arises because of the difference in geographical location of the observing site and the radiosonde station.

Figure 4 shows the results of measurements on a number of days at or near noon. The measurements were made with either the Golay cell or the indium antinomide-cooled detector. Since it is difficult to compare the absolute sensitivity of the two detectors with any precision by a laboratory experiment, we have compared the output trace heights at 1·3 mm, assuming an atmospheric absorption coefficient of 2·3 dB/km standard atmosphere at this wavelength. This figure for the attenuation at 1·3 mm is in agreement with previously published experimental data at this wavelength.

The spread of the points from a linear relation is much greater in figure 4 than in figure 3. While this may be due in part to changing experimental conditions from day to day, we think that the large deviations shown in figure 4 (r.m.s. fractional error in intensity $\sim$ 0·3) are mainly due to the differences between the actual water-vapour concentrations in the atmosphere above the telescope and that measured by the radiosonde 50 km away. The measurements shown in this figure give a value $6{\cdot}7 \pm 1{\cdot}0$ dB/km for an atmosphere containing 7·5 g per cubic metre of water vapour. This figure is in good if somewhat fortuitous agreement with that found from the results shown in figure 3. It is interesting to note that the difference between actual and radiosonde water-vapour concentrations produces an error which is systematic for the measurements of figure 3 but largely random for the results shown in figure 4. The error in these later results would of course be completely random if the water-vapour content of the atmosphere averaged over all days is the same above the telescope as it is at the radiosonde station. Comparison of radiosonde data from different stations indicates that this is a very good assumption.

Finally we consider three corrections which must be applied to the results:

(1) For a given amount of water-vapour, the decrease of atmospheric pressure with height produces a lower absorption than would be produced if all the water vapour were at ground level.

(2) The atmospheric temperatures are usually lower than those of a standard atmosphere (288 °K).

(3) The above analysis has been carried out for a monochromatic beam, whereas the window is relatively broad and the interferometer has a finite and in fact rather low spectral resolution (half width $\sim$ 0·5 cm^{-1}).

These three corrections produce fractional changes in the result of $+0{\cdot}06$, $-0{\cdot}10$, and $+0{\cdot}14$ respectively. The first two corrections were obtained on the assumption that the actual absorption coefficient varies with temperature and pressure in the same way as that calculated theoretically by Zhevakin & Naumov (1963). The third correction is slightly uncertain, since it is highly dependent on the resolving power of the interferometer and also on the optical thickness of the atmosphere. We have calculated the fractional correction which we quote here from the theoretical absorption spectra computed by Zhevakin & Naumov.

Using these corrections, we deduce a value of $7{\cdot}2 \pm 2{\cdot}0$ dB/km standard atmosphere. This result is somewhat lower than the value of 8·5 dB/km obtained by Ryadov, Furashov & Sharanov (1964) at a high altitude site.

4. Possibilities of submillimetre astronomy from low altitude sites

We shall now consider briefly the possibilities of submillimetre astronomy at sites either virtually at sea level or at altitudes which are sufficiently low for the physiological effects of height on the observer to be negligible. We shall examine the possibilities of measurement in five windows with minimum absorptions which occur at about 360, 460, 740, 860 and 1360 μm. Although it is possible to specify accurately the positions of absorption lines, the same is not true of the intervening windows. The wavelength at which the transmitted intensity is a maximum will depend on the wavelength dependence of the intensity of the source and the thickness, temperature, and pressure of the atmosphere. With the exception of the window at 1360 μm, these variables will probably produce variations of about 10 μm in the wavelength for maximum received intensity.

We shall ignore the window situated at about 620 μm, partly because the transmission is not particularly good and partly because the window is complicated by the presence of a number of fairly weak transitions (e.g. 5_4–6_2 and 4_4–5_0) which have absorption frequencies near the centre of the window.

Table 1. Atmospheric absorption

		360	460	740	860	1360
wavelength (μm)		360	460	740	860	1360
calculated absorption coefficient dB/km (standard atmosphere 760 mmHg, 288 °K, H_2O vapour 7·5 g/m)	King *et al.* (1947)	—	—	12·5	6·0	0·8
	Zhevakin & Naumov (1963)	44	38	13	6·0	1·3
	Bastin (1967)	43	45	12	5·3	1·0
mean experimental absorption coefficient ...		50 ± 5	50 ± 5	12 ± 4	$7·5 \pm 1·5$	$3·5 \pm 0·5$
source comparisons	precipitable water-vapour permissible (g)	0·04	0·04	0·12	0·20	0·4
	mean site temperature (°C) during coldest month should be less than	-40	-40	-20	-15	-3
contour mapping	precipitable water-vapour permissible (g)	0·2	0·2	0·6	1·0	2·0
	mean site temperature (°C) during coldest month should be less than	-15	-15	0	$+10$	$+20$

In table 1 we show values of the atmospheric attenuation in each of the windows considered; these values are obtained both by calculation and experiment. The theoretical values differ from each other mainly because they use different values of the line shape factor and specific half width. The experimental values given in the table are weighted means of previously reported experimental results. It is well known that about 1 mm the experimental attenuation values are about four times greater than the calculated values, and more recently it has been suggested by Bastin (1966) that this disparity becomes increasingly less marked in the submillimetre region. We should, however, point out that there are a number of measurements (Yaroslavsky & Stanevich 1959; Breeden, Rivers & Sheppard 1966) which suggest much higher attenuation in the 360 and 460 μm windows than that shown in table 1. We think that the very low optical thickness of the path-lengths used at the centres of the windows in these experiments is probably responsible for the apparently high-measured values of attenuation. Before considering possible sites for submillimetre astronomy, we examine two different types of observation which may be made:

(a) *Source comparisons*

In this type of observations we wish to measure the intensity of a point source by comparison with another source of known intensity situated in a different part of the sky. In this case because of the time taken for the telescope to move from one source to the other and also because of the possibility of different water-vapour concentrations in different angular directions, we think that the vertical atmospheric attenuation should not exceed about 2 dB (30 % loss of intensity).

(b) *Contour mapping*

The errors resulting from source comparisons are largely eliminated when a rapid scan is made across an extended source, provided of course that the only interest is in relative intensities within the source. In this case the main consideration is that the atmosphere transmits sufficient intensity for noise limitations of the detector not to become dominant. We set a somewhat arbitrary criterion of 10 dB for the vertical atmospheric attenuation (90 % loss of intensity) as the limit of this type of observation.

In this discussion we have considered vertical attenuations and have tried to take into account the increase in the slanted water-vapour path-length in any actual observation by making the criteria somewhat severe. This procedure is of course only very approximate, and more stringent criteria are necessary for sources with high zenith angles. The precise criteria which should be adopted will also of course depend on the accuracy required in the final results.

Table 1 shows the limiting precipitable water vapour for both types of observation as well as typical surface temperatures which are associated with this amount of precipitable water vapour.

In considering possible sites for submillimetre observation, we have concluded that in general the low temperature of the site is perhaps the most important consideration. This is shown clearly in table 2. Here sites have been chosen from the equator to the poles at about 10° intervals of latitude but at dry sites and, with the exception of the south pole site, all below 2000 m altitude. A change of a factor of about 200 in the precipitable water vapour between the pole and equatorial sites seems to us very important. By contrast arid sites do not seem to have precipitable water-vapour contents which are much lower than humid sites at the same temperature. The advantage of a high altitude site is more marked. However, to produce a decrease of a factor of 50 in the water-vapour content, the site normally needs to be at an elevation of 6–7 km at which physiological effects become very marked.

We conclude this discussion with the following points:

(1) Many of the world's observatories could make observations in the 730 and 860 μm windows for a limited period of each year.

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(2) The temperature of a site is as important as its elevation in determining its suitability for submillimetre measurements. Measurements in the 360 and 460 μm windows are almost certainly possible in cold climates without the need of a high altitude site.

(3) The possibilities of using the Amundsen Scott station at the south pole as a site could well be investigated. The calculations of Zhevakin and Naumov suggest that measurements in the range 30–300 μm would also be possible from this station.

Table 2. Precipitable water vapour and attenuation data for sites at different latitudes

In each case the figures refer to mean values during the driest month.

site	latitude	longitude	precipitable water vapour (g/cm^2)	month	attenuation in vertical path through atmosphere (dB/km) 360 μm	460 μm	740 μm	860 μm	1360 μm
Amundsen–Scott Station, South Pole	90° 00′ S	—	0·012	July	0·62	0·62	0·2	0·15	0·06
Eureka, Canada	80° 00′ N	85° 56′ W	0·080	Jan.	5·4	5·4	1·4	1·1	0·4
Hall Beach, Canada	68° 47′ N	81° 15′ W	0·080	Jan.	5·4	5·4	1·4	1·1	0·4
Seimchan, U.S.S.R.	62° 55′ N	152° 25′ E	0·138	Jan.	8·2	8·2	2·4	1·8	0·7
Blagoveshchensk, U.S.S.R.	50° 16′ N	127° 30′ E	0·24	Jan.	16	16	4·3	3·3	1·2
Grand Junction, Colo., U.S.A.	39° 06′ N	108° 32′ W	0·55	Jan.	37	37	10	7·5	2·6
Colomb-Bechar, Algeria	31° 38′ N	02° 15′ W	0·90	Jan.	60	60	16	12	4·4
Niamey, Niger Republic	13° 29′ N	02° 10′ E	1·45	Jan.	97	97	26	20	7·1
Niarobi, Kenya	01° 18′ S	36° 45′ E	3·4	Aug.	220	220	60	45	16

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